Head Office: 2nd Floor, Grand Plaza, Fraser Road, Dak Bunglow, Patna - 01

JEE Main 2023 (Memory based)

30 January 2023 - Shift 2

Answer & Solutions

MATHEMATICS

22. Common tangent is drawn to $y^2 = 16x$ and $x^2 + y^2 = 8$. The square of distance between point of contact of common tangent on both the curves is:

A. 78
B. 72
C. 42
D. 76

Answer (B)
Solution:

Given equation of parabola is $y^2 = 16x$

$\Rightarrow a = 4$

General equation of tangent to a parabola is $y = mx + \frac{a}{m}$

Given equation of circle is $x^2 + y^2 = 8$

$\Rightarrow r = \sqrt{8}$ and Centre of circle $(0,0)$

$y = mx + \frac{4}{m}$ is tangent to circle

$\therefore$ Perpendicular distance from $(0,0)$ to $y = mx + \frac{4}{m}$ is equal to radius of circle.

$\Rightarrow \left|\frac{\frac{4}{m}}{\sqrt{m^2+1}}\right| = \sqrt{8}$

$\Rightarrow \frac{16}{m^2} = 8m^2 + 8$

$\Rightarrow 8m^4 + 8m^2 - 16 = 0$

$\Rightarrow 8m^4 + 16m^2 - 8m^2 - 16 = 0$

$\Rightarrow 8m^2(m^2 + 2) - 8(m^2 + 2) = 0$

$\Rightarrow m = \pm 1$

Point of contact on parabola $= \left(\frac{a}{m^2}, \frac{2a}{m}\right)$

$= (4, \pm 8)$

Point of contact on circle $= \left(\pm\frac{am}{\sqrt{1+m^2}}, \mp\frac{a}{\sqrt{1+m^2}}\right)$

$= (-2, +2)$ or $(-2, -2)$

Distance between $(4,8)$ and $(-2, 2) = \sqrt{6^2 + 6^2} = \sqrt{72}$

Also, Distance between $(4, -8)$ and $(-2, -2) = \sqrt{6^2 + 6^2} = \sqrt{72}$

$\therefore$ Square of distance between point of contact of common tangent on both the curves $= 72$

23. Let $f(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 1, & x = 0 \end{cases}$, $g(x) = \begin{cases} \frac{\sin(x+1)}{x+1}, & x \neq -1 \\ 1, & x = -1 \end{cases}$, $h(x) = 2[x] + f(x)$ ($[.]$ denotes greatest integer function).Then $\lim_{x\to 1} g(h(x-1))$ is :

A. $\frac{\sin 1}{1}$

B. $\frac{\sin 2}{2}$

C. -1

D. 2

Answer (B)

Solution:

$h(x-1) = 2[x-1] + f(x-1)$

$\lim_{x \to 1^+} h(x-1) = 2 \cdot 0 + f(0^+) = 1$

$\lim_{x \to 1^-} h(x-1) = 2 \cdot (-1) + f(0^-) = 2 \cdot (-1) + (-1) = -3$

R.H.L.: $\lim_{x \to 1^+} g(h(x-1)) = g(1) = \frac{\sin(1+1)}{1+1} = \frac{\sin 2}{2}$

L.H.L.: $\lim_{x \to 1^-} g(h(x-1)) = g(-3) = \frac{\sin(-3+1)}{(-3+1)} = \frac{\sin 2}{2}$

$\Rightarrow$ L.H.L.= R.H.L.

$\therefore \lim_{x \to 1} g(h(x-1)) = \frac{\sin 2}{2}$

24. If $|\vec{a}| = 1,\ |\vec{b}| = 2,\ \vec{a} \cdot \vec{b} = 4,\ \vec{c} = 2(\vec{a} \times \vec{b}) - 3\vec{b}$. Then $\vec{b} \cdot \vec{c}$ equals:

A. -48
B. -12
C. 12
D. 48

Answer (B)

Solution:

$\vec{c} = 2(\vec{a} \times \vec{b}) - 3\vec{b}$

$\vec{b} \cdot \vec{c} = 2\vec{b} \cdot (\vec{a} \times \vec{b}) - 3|\vec{b}|^2$

$\vec{b} \cdot \vec{c} = -3|\vec{b}|^2$ $\cdots$ (since $(\vec{a} \times \vec{b}) \cdot \vec{b} = 0$)

$\vec{b} \cdot \vec{c} = -12$

25. $\lim_{n \to \infty} \frac{3}{n}\left[4 + \left(2 + \frac{1}{n}\right)^2 + \left(2 + \frac{2}{n}\right)^2 + \cdots + \left(3 - \frac{1}{n}\right)^2\right]$ is:

A. 19
B. 21
C. -19
D. 0

Answer (A)

Solution:

$\lim_{n \to \infty} \frac{3}{n}\left[4 + \left(2 + \frac{1}{n}\right)^2 + \left(2 + \frac{2}{n}\right)^2 + \cdots + \left(3 - \frac{1}{n}\right)^2\right]$ $\cdots$ (given)

we can rewrite the above equation as

$$\lim_{n\to\infty}\frac{3}{n}\left[\left(2+\frac{0}{n}\right)^2+\left(2+\frac{1}{n}\right)^2+\left(2+\frac{2}{n}\right)^2+\cdots+\left(2+\left(\frac{n-1}{n}\right)\right)^2\right]$$

$$\Rightarrow \lim_{n\to\infty}\frac{3}{n}\sum_{r=0}^{n-1}\left(2+\frac{r}{n}\right)^2$$

$$\frac{r}{n}\to x$$

$$\frac{1}{n}\to dx$$

$$\frac{0}{n}<\frac{r}{n}<\frac{n-1}{n}$$

$$\Rightarrow 0<\frac{r}{n}<1-\frac{1}{n}$$

$$\Rightarrow \lim_{n\to\infty}0<\lim_{n\to\infty}\frac{r}{n}<\lim_{n\to\infty}1-\frac{1}{n}$$

$$\Rightarrow 0<\lim_{n\to\infty}\frac{r}{n}<1$$

$$\Rightarrow \lim_{n\to\infty}\frac{3}{n}\sum_{r=0}^{n-1}\left(2+\frac{r}{n}\right)^2 = 3\int_0^1(2+x)^2\,dx$$

$$= 3\cdot\left[\frac{(2+x)^3}{3}\right]_0^1$$

$$= 27-8$$

$$\Rightarrow \lim_{n\to\infty}\frac{3}{n}\sum_{r=0}^{n-1}\left(2+\frac{r}{n}\right)^2 = 19$$

26. Let $f(x)=\sqrt{3-x}+\sqrt{x+2}$. The range of $f(x)$ is:

A. $(2\sqrt{2},\sqrt{10})$
B. $(\sqrt{5},\sqrt{10})$
C. $(\sqrt{2},\sqrt{7})$
D. $(\sqrt{7},\sqrt{10})$

Answer (B)

Solution:

$$y=\sqrt{3-x}+\sqrt{x+2}$$

$$y'=\frac{1}{2\sqrt{3-x}}(-1)+\frac{1}{2\sqrt{x+2}}=0$$

$$\Rightarrow \sqrt{3-x}=\sqrt{x+2}$$

$$\Rightarrow x=\frac{1}{2}$$

$$y\left(\frac{1}{2}\right)=\sqrt{\frac{5}{2}}+\sqrt{\frac{5}{2}}$$

$$y_{\max}=\sqrt{10}$$

$y_{\min}$ at $x=-2$ or $x=3=\sqrt{5}$

$$\therefore y\in[\sqrt{5},\sqrt{10}]$$

27. The value of $\tan^{-1}\left(\frac{1}{1+a_1a_2}\right)+\tan^{-1}\left(\frac{1}{1+a_2a_3}\right)+\cdots+\tan^{-1}\left(\frac{1}{1+a_{2021}a_{2022}}\right)$. If $a_1=1$ and a_i are consecutive natural numbers

A. $\frac{\pi}{4} - \cot^{-1} 2021$
B. $\frac{\pi}{4} - \cot^{-1} 2022$
C. $\frac{\pi}{4} - \tan^{-1} 2021$
D. $\frac{\pi}{4} - \tan^{-1} 2022$

Answer (B)

Solution:

$\tan^{-1}\left(\frac{a_2-a_1}{1+a_1a_2}\right) + \tan^{-1}\left(\frac{a_3-a_2}{1+a_2a_3}\right) + \cdots + \tan^{-1}\left(\frac{a_{2022}-a_{2021}}{1+a_{2021}a_{2022}}\right).$

$= (\tan^{-1} a_2 - \tan^{-1} a_1) + (\tan^{-1} a_3 - \tan^{-1} a_2) + \cdots + (\tan^{-1} a_{2022} - \tan^{-1} a_{2021})$

$= \tan^{-1}(a_{2022}) - \tan^{-1}(a_1))$

As $a_1 = 1, a_2 = 2 \cdots, a_{2022} = 2022$

$= \tan^{-1}(2022) - \tan^{-1}(1))$

$= \tan^{-1}(2022) - \frac{\pi}{4}$

$= \frac{\pi}{2} - \cot^{-1}(2022) - \frac{\pi}{4}$

$= \frac{\pi}{4} - \cot^{-1}(2022)$

28. Let $P = \left(8\sqrt{3} + 13\right)^{13}$, $Q = \left(6\sqrt{2} + 9\right)^{9}$ then : (where $[.]$ represents G.I.F.)

A. $[P]$ = odd, $[Q]$ = even
B. $[P] =$ even, $[Q] =$ odd
C. $[P] =$ odd, $[Q] =$ odd
D. $[P] + [Q]$ = even

Answer (B)

Solution:

Let $P = I + f_1 = \left(8\sqrt{3} + 13\right)^{13}$ such that $(0 < f_1 < 1)$ and $Let\ f_1' = \left(8\sqrt{3} - 13\right)^{13}$ such that $(0 < f_1{}' < 1)$.

$I + f_1 - f_1' = \left(8\sqrt{3} + 13\right)^{13} - \left(8\sqrt{3} - 13\right)^{13}$

$= 2\left({}^{13}C_1\left(8\sqrt{3}\right)^{12}(13)^1 + {}^{13}C_3\left(8\sqrt{3}\right)^{10}(13)^3 + {}^{13}C_5\left(8\sqrt{3}\right)^{8}(13)^5 + \cdots + {}^{13}C_{13}\left(8\sqrt{3}\right)^{0}(13)^{13}\right)$

$I + f_1 - f_1' = 2p$

Since $-1 < f_1 - f_1' < 1 \Rightarrow f_1 - f_1' = 0$

$\Rightarrow I_1 = 2p$

So, I_1 is even.

Let $Q = I_2 + f_2$ such that $(0 < f_2 < 1)$

Also, let $f_2' = \left(9 - 6\sqrt{2}\right)^{9}$

$I_2 + f_2 + f_2' = \left(9 + 6\sqrt{2}\right)^{9} + \left(9 - 6\sqrt{2}\right)^{9}$

$= 2\left({}^{9}C_0(9)^9\left(6\sqrt{2}\right)^{0} + {}^{9}C_2(9)^7\left(6\sqrt{2}\right)^{2} + {}^{9}C_4(9)^5\left(6\sqrt{2}\right)^{4} + \cdots + {}^{9}C_8(9)^1\left(6\sqrt{2}\right)^{8}\right)$

$I_2 + f_2 + f_2' = 2p$ where $p \in Z$

$0 < f_2 + f_2' < 2$

$f_2 + f_2' = 1$
$I_2 + 1 = 2p$
$\Rightarrow I_2 = 2p - 1$
$\Rightarrow [Q]=$ odd number

29. Let p: I am well.,
q: I will not take rest.
r: I will not sleep properly,
then "If I am not well then I will not take rest and I will not sleep properly" is logically equivalent to:

A. $(\sim p \rightarrow q) \vee r$
B. $\sim p \rightarrow (q \wedge r)$
C. $(\sim p \wedge q) \rightarrow r$
D. $(\sim p \vee q) \rightarrow r$

Answer (B)

Solution:

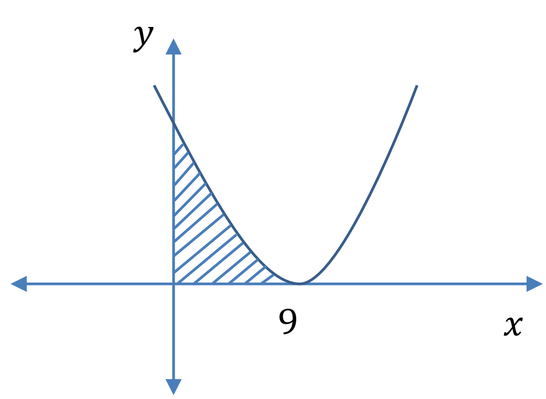
$\sim p$: I am not well
q: I will not take rest.
r: I will not sleep properly
I will not take rest and I will not sleep properly $\equiv q \wedge r$
If I am not well then I will not take rest and I will not sleep properly $\equiv \sim p \rightarrow (q \wedge r)$

30. q is maximum value of P lying in interval $[0, 10]$, roots of $x^2 - Px + \frac{5P}{4} = 0$ are having rational roots. Find area of region $S: \{0 \leq y \leq (x-q)^2\}$

A. 243
B. 723
C. 81
D. 3

Answer (A)

Solution:



$D = P^2 - 5P$ must be perfect square i.e possible when $P = 9$
Region for $0 \leq y \leq (x-9)^2$ is in 1st quadrant
$A = \int_0^9 (x-9)^2 dx$
$A = \left[\frac{(x-9)^3}{3}\right]_0^9$
$A = 0 + \frac{9^3}{3} = 243$ sq. unit

31. If $\frac{dy}{dx} = -\frac{3x^2+y^2}{3y^2+x^2}$, $y(1) = 0$ then $f(x)$ is:

A. $\log(x+y)+\frac{2xy}{(x+y)^2}=0$

B. $\log(x+y)-\frac{2xy}{(x+y)^2}=0$

C. $3=(3y^2-2xy+3x^2)(x+y)^2$

D. $3=(3y^2-2xy+3x^2)(x+y)$

Answer (A)

Solution:

$\frac{dy}{dx}=-\frac{3x^2+y^2}{3y^2+x^2}=-\frac{3+\left(\frac{y}{x}\right)^2}{3\left(\frac{y}{x}\right)^2+1}$

Let $\frac{y}{x}=u$

$\frac{dy}{dx}=u+x\frac{du}{dx}$

$u+x\frac{du}{dx}=\frac{-(3+u^2)}{3u^2+1}$

$x\frac{du}{dx}=\frac{-(3+u^2)-u(3u^2+1)}{3u^2+1}$

$x\frac{du}{dx}=\frac{-[3u^3+u^2+u+3]}{(3u^2+1)}$

$x\frac{du}{dx}=\frac{-(u+1)(3u^2-2u+3)}{(3u^2+1)}$

$\int\frac{(3u^2+1)}{(u+1)(3u^2-2u+3)}du=-\int\frac{dx}{x}$

$\int\left(\frac{\frac{1}{2}}{u+1}+\frac{\frac{1}{4}(6u-2)}{3u^2-2u+3}\right)du=-\int\frac{dx}{x}$

$\frac{1}{2}\ln(x+y)-\frac{1}{2}\ln x+\frac{1}{4}\ln(3y^2-2xy+3x^2)-\frac{1}{4}\times 2\ln x=-\ln x+C$

$\ln(x+y)^2+\ln(3y^2-2xy+3x^2)=C$

$(x+y)^2(3x^2-2xy+3y^2)=C$

$y(1)=0\Rightarrow C=3$

$(x+y)^2(3x^2-2xy+3y^2)=3$

32. Two A.P's are given as under $3,\ 7,\ 11,\ \cdots$ and $1,\ 6,\ 11,\ 16,\ \cdots$Then 8^{th} common term that is appearing in both the series is __________.

Answer (151)

Solution:

First common term is 11 and common terms will appear in an A.P having common difference as LCM of $(4,\ 5)=20$

$T_8=11+(8-1)20$

$T_8=151$

33. Using the digits $1,\ 2,\ 2,\ 2,\ 3,\ 3,\ 5$ number of 7-digit odd numbers that can be formed is _________.

Answer (240)

Solution:

We need 7-digit odd numbers,

Hence the unit digit will any one of $\{1, 3\ 5\}$

_ _ _ _ _ _ 1

Total numbers with unit digit $1=\frac{6!}{2!3!}=60$

_ _ _ _ _ _ 3

Total numbers with unit digit $3=\frac{6!}{3!}=120$

_ _ _ _ _ _ 5

Total numbers with unit digit 5 $= \frac{6!}{3!2!} = 60$
Total 7- digit odd numbers $= 60 + 120 + 60 = 240$

34. 50^{th} Root of x is 12.
50^{th} Root of y is 18.
Reminder when $x + y$ is divided by 25 is _________.

Answer (23)

Solution:

$x + y = 12^{50} + 18^{50} = 144^{25} + 324^{25}$
$= (25k_1 - 6)^{25} + (25k_2 - 1)^{25}$
$= 25\lambda - 6^{25} - 1$
$6^{25} + 1 = (6^5)^5 + 1$
$= (7776)^5 + 1$
$= (25\lambda_1 + 1)^5 + 1$
$= 25p + 2$
$\Rightarrow 12^{50} + 18^{50} = 25\lambda - (25p + 2) = 25\lambda - 25p - 2 = 25\lambda - 25p - 25 + 23 = 25n + 23 \; where \; n = \lambda - p - 1$
$\Rightarrow$ Reminder$= 23$

35. Let $a = \{1,3,5, \ldots ,99\}$ & $b = \{2,4,6, \ldots ,100\}$ The number of ordered pairs (a, b) such that $a + b$ when divided by 23 leaves remainder by 23 leaves remainder 2 is ______.

Answer (109)

Solution:

$a + b = 23\lambda + 2$ where $\lambda = 0,1,2, \ldots$
but λ can't be even. So, if
if $\lambda = 1$ (a, b) $\rightarrow 12$ pairs
if $\lambda = 3$ (a, b) $\rightarrow 35$ pairs
if $\lambda = 5$ (a, b) $\rightarrow 42$ pairs
if $\lambda = 7$ (a, b) $\rightarrow 19$ pairs
if $\lambda = 9$ (a, b) $\rightarrow 0$ pairs
Total $= 12 + 35 + 42 + 19 = 108$ ordered pairs

36. If area of the region bounded by the curves $y = x^2$, $y = (1 - x)^2$ and $y = 2x(1 - x)$ is A, then the value of $540A$ is __________.

Answer (135)

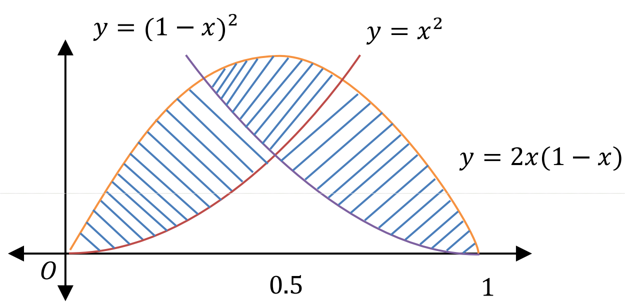
Solution:

$A = \int_0^1 2x(1 - x)dx - \int_0^{\frac{1}{2}} x^2 dx - \int_{\frac{1}{2}}^1 (1 - x)^2 dx$

$= \left[x^2 - \frac{2x^2}{3}\right]_0^1 - \left[\frac{x^3}{3}\right]_0^{\frac{1}{2}} + \left[\frac{(1-x)^3}{3}\right]_{\frac{1}{2}}^1$

$= \frac{1}{4}$

$\Rightarrow 540A = 540 \times \frac{1}{4} = 135$



37. $A = \{2,4,6,8,10\}$ Then the total no of functions defined on A such that $F(m \cdot n) = F(m) \cdot F(n)$, $m, n \in A$ are _________.

Answer (25)

Solution:

$f(\mathrm{m} \cdot n) = f(m) \cdot f(n),\ m,\ n \in A$
$f(x) = x^k, k \in R$
$f(2) = 2^k$ can be connected to 5 objects
$f(4) = 4^k$ can be connected to 5 objects
$f(6) = 6^k$ can be connected to 5 objects
$f(8) = 8^k$ can be connected to 5 objects
$f(10) = 10^k$ can be connected to 5 objects
Total functions= $5 \times 5 = 25$